MATHEMATICS METHODS

MAWA Semester 1 (Unit 3) Examination 2019
Calculator-assumed

Marking Key

The release date for this exam and marking scheme is 14th June.

Section Two: Calculator-assumed (100 Marks)

Question 8(a) (4 marks)

Solution

$$\int_2^6 \frac{f(x)}{3}dx = 4$$

(i) $\int_2^6 f(x)dx = 3\times 4 = 12$

(ii) $\int_2^6 \frac{3f(x)-1}{2}dx = \int_2^6 \frac{3f(x)}{2}dx - \int_2^6 \frac{1}{2}dx$

$= \frac{3}{2}\int_2^6 f(x)dx - \int_2^6 \frac{1}{2}dx$

$= \left(\frac{3}{2}\times 12\right) - \left[\frac{1}{2}x\right]_2^6$

$= 18 - [3-1]$

$= 16$

Mathematical behaviours	Marks
• states $\int_2^6 f(x)dx = 12$	1
• uses linearity and additivity to deduce $\int_2^6 \frac{3f(x)-1}{2}dx = \int_2^6 \frac{3f(x)}{2}dx - \int_2^6 \frac{1}{2}dx$	1
• anti-differentiates $\frac{1}{2}$	1
• determines correct result of 16	1

Question 8(b) (3 marks)

Solution

$\int_{-\frac{1}{4}}^{0} e^{4x+1}dx = \frac{1}{4}\int_{-\frac{1}{4}}^{0} 4e^{4x+1}dx$

$= \frac{1}{4}\left[e^{4x+1}\right]_{-\frac{1}{4}}^{0}$

$= \frac{1}{4}\left[e^{1} - e^{0}\right]$

$= \frac{1}{4}[e-1]$

Mathematical behaviours	Marks
• anti-differentiates correctly	1
• substitutes limits of integration correctly	1
• determines exact result	1

Question 9(a) **(1 mark)**

Solution	
$f'(x)=(x-1)^2(4x-1)=4x^3+bx^2+cx+d+e$ $hence\ f(x)=x^4+....\qquad ie\quad a>0$	
Mathematical behaviours	Mark
• states $a>0$ justifies answer using anti-differentiation	1

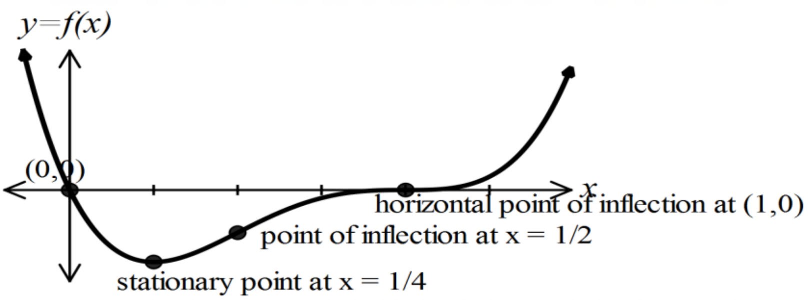
Question 9(b) **(1 mark)**

Solution	
For stationary points, $f'(x)=0$ $ie\,(x-1)^2(4x-1)=0 \Rightarrow x=1,\frac{1}{4}$	
Mathematical behaviours	Marks
• states x coordinates of stationary points	1

Question 9(c) **(3 marks)**

Solution	
$f'(1)=0$ and $f''(1)=0$ $f''(x)=6(x-1)(2x-1)$ $f''(1^-)=-ve\times+ve=-ve$ $f''(1^+)=+ve\times+ve=+ve$ Hence there is a change in concavity at $x=1$ and $f'(1)=0$ so there is a horizontal point of inflection at $x=1$. Hence *m* = 1.	
Mathematical behaviours	Marks
• states $f'(1)=0$ and $f''(1)=0$ • demonstrates change in concavity at $x=1$ • states that horizontal point of inflection occurs at *m* = 1.	1 1 1

Question 9(d) **(3 marks)**

Solution	
	
Mathematical behaviours	Marks
• sketch shows $x\to\pm\infty,\ f(x)\to\infty$ and roots • clearly shows x coordinate of minimum turning point • graphs correct shape and clearly labels points of inflection	1 1 1

Question 10(a) **(1 mark)**

Solution	
X has a binomial distribution with parameters n and $p = 0.5$ ie $X \sim Bin(n, 0.5)$	
Mathematical behaviours	Marks
• identifies binomial distribution and states parameters	1

Question 10(b) **(1 mark)**

Solution	
$E(X) = \mu = np = \frac{n}{2}$	
Mathematical behaviours	Marks
• states correct answer	1

Question 10(c) **(3 marks)**

Solution	
$n = 20$: $P_1 = \mathrm{P}(5 \le X \le 15) \cong 0.988$ $n = 1000$: $P_1 = \mathrm{P}(495 \le X \le 505) \cong 0.272$ $n = 10\,000$: $P_1 = \mathrm{P}(4995 \le X \le 5005) \cong 0.088$ (from calculator)	
Mathematical behaviours	Marks
• states a probability inequality relevant to one of the n values • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(d) **(1 mark)**

Solution	
$P_1 \to 0$ as $n \to \infty$	
Mathematical behaviours	Marks
• obtains correct answer	1

Question 10(e) **(3 marks)**

Solution	
$n = 20$: $P_2 = \mathrm{P}(9.5 \le X \le 10.5) = P(X = 10) \cong 0.176$ $n = 200$: $P_2 = \mathrm{P}(95 \le X \le 105) \cong 0.563$ $n = 1000$: $P_2 = \mathrm{P}(475 \le X \le 525) \cong 0.893$	
Mathematical behaviours	Marks
• states probability inequality relevant to $n = 20$ • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(f) **(1 mark)**

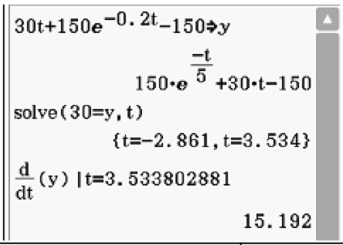
Solution	
$P_2 \to 1$ as $n \to \infty$	
Mathematical behaviours	Marks
• obtains correct answer	1

Question 11(a) **(1 mark)**

Solution	
$y = 30t + 150e^{-0.2t} + k$ $t = 0, y = 0 \Rightarrow 0 = 150 + k \Rightarrow k = -150$	
Mathematical behaviours	**Mark**
• evaluates k	1

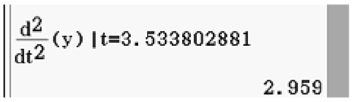
Question 11(b) **(3 marks)**

Solution	
$y = 30t + 150e^{-0.2t} - 150$ $y = 30 \Rightarrow t = 3.53s$ $v = 30 - 30e^{-0.2t}$ $v\big\|_{t=3.53} = 15.19m/s$	
Mathematical behaviours	**Marks**
• equates $y = 30$ and determines time taken to hit the ground	1
• differentiates to obtain v	1
• calculates the speed	1



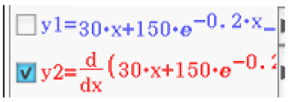
Question 11(c) **(2 marks)**

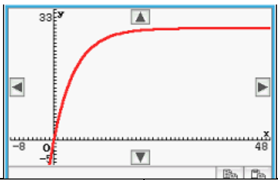
Solution	
$v = 30 - 30e^{-0.2t}$ $\Rightarrow a = 6e^{-0.2t} m/s^2 > 0$ Since $v > 0$ and $a > 0$ the ball is speeding up.	
Mathematical behaviours	**Marks**
• differentiates v to determine a and states $a > 0$	1
• draws conclusion noting the same sign of both v and a.	1



Question 11(d) **(1 mark)**

Solution	
$v = 30 - 30e^{-0.2t}$, $a = 6e^{-0.2t}$ $t \to \infty, v \to 30, a \to 0$ Hence constant speed is attained.	
Mathematical behaviours	**Marks**
• states $v \to 30\,m/s$ ie is constant	1





Question 11(e) **(1 mark)**

Solution	
A restriction on the domain is needed. ie $0 \le t \le 3.53$	
Mathematical behaviours	**Marks**
• states restriction required on the domain	1

Question 12(a) **(2 marks)**

Solution	
$\mu = \frac{49\times 63.3+38\times 54.1}{87} = 59.28$	
Mathematical behaviours	Marks
• uses correct expression • obtains correct answer	1 1

Question 12(b) **(4 marks)**

Solution	
If $Y = aX + b,$ then $E(Y) = aE(X) + b$ and $St.Dev(Y) = aSt.Dev(X)$ So $59.28 = a \times 63.3 + b$ and $9 = a \times 7.6$ So $a = 1.18$ and $b = -15.68$	
Mathematical behaviours	Marks
• expresses $E(Y)$ in terms of $E(X)$ • expresses $Std\ Dev\ (Y)$ in terms of $Std\ Dev(X)$ • calculates a • calculates b	1 1 1 1

Question 13(a) **(5 marks)**

Solution

Stationary Points: $\frac{dy}{dx}=0$

i.e. $(6x-1)\left(x+\frac{1}{2}\right)=0$

$x=\frac{1}{6}$ or $x=-\frac{1}{2}$

Now $\frac{dy}{dx}=(6x-1)\left(x+\frac{1}{2}\right)$

$= 6x^2+2x-\frac{1}{2}$

$\frac{d^2y}{dx^2}=12x+2$

At $x=\frac{1}{6}, \frac{d^2y}{dx^2}=4 \Rightarrow \text{min}$ At $x=-\frac{1}{2}, \frac{d^2y}{dx^2}=-4 \Rightarrow \text{max}$

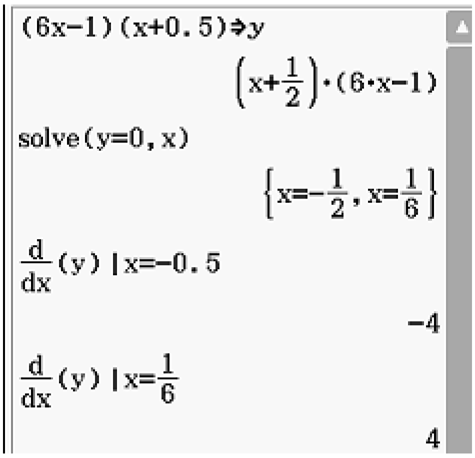
$\therefore$ max turning pt at $\left(-\frac{1}{2},1\right)$

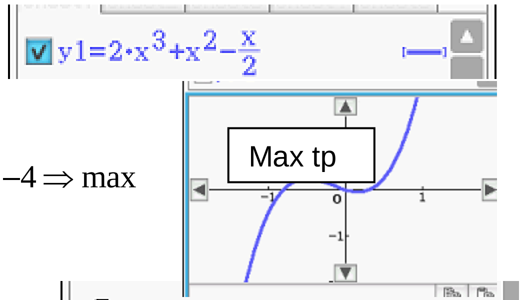
Now $y=2x^3+x^2-\frac{1}{2}x+c$

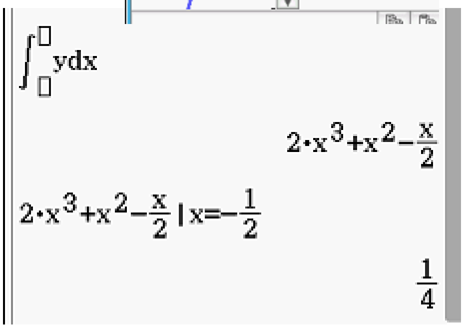
$\left(-\frac{1}{2},1\right) \Rightarrow 1=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-\frac{1}{2}.\left(-\frac{1}{2}\right)+c$

$1=\frac{1}{4}+c \Rightarrow c=\frac{3}{4}$

$\therefore$ equation of the function is $y=2x^3+x^2-\frac{1}{2}x+\frac{3}{4}$







Mathematical behaviours	Marks
• uses $\frac{dy}{dx}=0$ to find stationary points	1
• substitutes into $\frac{d^2y}{dx^2}$, $x=\frac{1}{6}$ and $x=-\frac{1}{2}$ to find which x value gives a local maximum turning point or clearly shows on sketch location of maximum and confirms maximum using 2nd derivative test	1 1
• integrates the derivative function correctly	1
• uses the point $\left(-\frac{1}{2},1\right)$ to determine the value of c	1
• states the correct equation of the function	1

Question 13(b) **(5 marks)**

Solution
(i) $V = \frac{\pi h}{3}(R^2 + r^2 + Rr)$ $V = \frac{\pi(15)}{3}(5^2 + 3^2 + 5\times 3)$ $\approx 769.69\,cm^3 \approx 770\,cm^3$ (ii) $V = \frac{\pi 15}{3}(R^2 + 3^2 + 3R)$ $\frac{dV}{dR} = 5\pi(2R+3)$ $\frac{dV}{dR} \approx \frac{\delta V}{\delta R}, R = 5, \delta R = -0.2$ $\delta V \approx 5\pi(2\times 5 + 3)(-0.2)$ $\delta V \approx -40.84\ cm^3 \approx -41\ cm^3$ *ie a decrease in capacity of approximately* 41 *millilitres*

Mathematical behaviours	Mark
(i)	
• states correct volume to the nearest cubic centimetre	1
(ii)	
• states V in terms of R	1
• uses incremental formula to obtain expression for small change in V	1
• substitutes, $R = 5$ and $\delta R = -0.2$	1
• states the decrease in capacity	1

Question 14(a) **(3 marks)**

Solution

Total number of cars in sample is $27 + 13 + 11 + 4 + 14 = 69$

Proportions of the various colours, and rounded to a whole multiple of 0.05:

White	Black	Red	Blue	Other
$\frac{27}{69} \cong 0.391$ $\cong 0.4$	$\frac{13}{69} \cong 0.188$ $\cong 0.2$	$11/69$ $\cong 0.159$ $\cong 0.15$	$4/69 \cong$ $0.0580.0 \cong$ 0.07	$14/69$ $\cong 0.203$ $\cong 0.2$

Mathematical behaviours	Marks
• obtains total sample size	1
• calculates all fractions correctly	1
• rounds all answers correctly	1

Question 14(b) **(3 marks)**

Solution	
Expected number of points per car $= 2 \times 0.4 + 4 \times 0.2 + 7 \times 0.15 + 9 \times 0.05 + 5 \times 0.2 = 4.1$ So expected number of points per 100 cars $= 100 \times 4.1 = 410$	
Mathematical behaviours	Marks
• obtains correct expression for expected value • calculates expected value (per car) correctly • obtains correct answer	1 1 1

Question 14(c) **(2 marks)**

Solution

Expected number of points per car (by colour)

White	Black	Red	Blue	Other
$2 \times 0.4 = 0.8$	$4 \times 0.2 = 0.8$	$7 \times 0.15 = 1.05$	$9 \times 0.05 = 0.45$	$5 \times 0.2 = 1$

Since the expected points per car is greatest for Rodney's red cars, Rodney is most likely to accumulate points fastest.

Mathematical behaviours	Marks
• evaluates expected values correctly • correct answer	1 1

Question 14(d) **(2 marks)**

Solution	
$P = 0.4^2 + 0.2^2 + 0.15^2 + 0.05^2 + 0.2^2 = 0.265$	
Mathematical behaviours	Marks
• uses correct formula • evaluates correctly	1 1

Question 15(a) **(4 marks)**

Solution	
(i) none (consecutive selections are not independent so not binomial) (ii) uniform (iii) binomial (iv) binomial	
Mathematical behaviours	Marks
i) • states none (ii) • states uniform (iii) • states binomial (iv) • states binomial	 1 1 1 1

Question 15(b) **(4 marks)**

Solution	
(i) $f(x)=\frac{x}{6}$, where $x=-1,1,2,4$. No, since $f(-1)=\frac{-1}{6}$ represents a probability and probability cannot be negative	
(ii) see table below. Yes as $0 \le f(x) \le 1 \ \forall \ x$ and the sum of the probabilities is 1.	
Mathematical behaviours	**Marks**
(i)	
• states no	1
• recognises negative probability	1
(ii)	
• states yes	1
• states both reasons	1

(ii)

x	4	6	8	10
$f(x)$	0.05	0.30	0.25	0.4

Question 16(a) **(3 marks)**

Solution

$v=\int 8\,dt$

$v=\frac{ds}{dt}=8t+c$

$v(0)=p \Rightarrow \quad p=c$

$\therefore v=\frac{ds}{dt}=8t+p$

$s=4t^2+pt+k$

$s(1)=q \Rightarrow \quad q=4+p+k \ \Rightarrow \quad k=q-4-p$

$\therefore s=4t^2+pt+q-4-p$

or $s=4t^2+pt+q-p-4$ as required

Mathematical behaviours	Marks
• anti-differentiates $a(t)$ to obtain $v(t)$ and uses $v(0)=p$ to get correct expression for c.	1
• anti-differentiates $v(t)$ to obtain $s(t)$ and uses $s(1)=q$ to get correct expression for k	1
• states required answer	1

Question 16(b) **(1 mark)**

Solution	
Distance travelled = $\int_0^3 \lvert 8t+p \rvert dt$	
Mathematical behaviours	Marks
• states the integral of the **absolute** velocity function from $t=0$ to $t=3$	1

Question 17(a) **(2 marks)**

Solution	
Define the random variable, X as the number of batteries that last for less than 2000 hours. Hence, $X \sim Bin(120, 0.1)$ $P(X=15) \approx 0.0742$	
Mathematical behaviours	Marks
• recognizes Binomial nature • obtains correct answer	1 1

Question 17(b) **(2 marks)**

Solution	
$X \sim Bin(120, 0.1)$ $P(X \le 15) \approx 0.8560$	
Mathematical behaviours	Marks
• recognizes binomial nature • obtains correct answer	1 1

Question 17(c) **(2 marks)**

Solution	
From part (b) we can conclude that there is an 85.6% chance that no more than 15 batteries out of 120 last less than 2000hrs. This would imply that there is only a 14.4% chance that more than 15 out of 120 batteries last less than 2000hrs. Hence the test does not imply compelling evidence that the manufacturer's claim is false.	
Mathematical behaviours	Marks
• obtains correct answer • gives valid reason	1 1

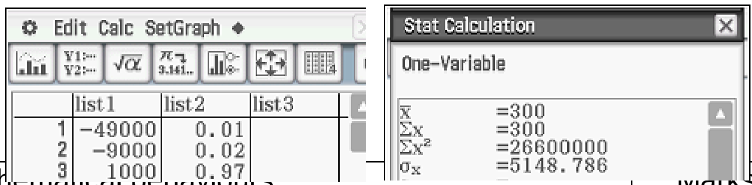
Question 18(a) **(3 marks)**

Solution

Outcome	Death	Permanent Disability	No payout
Profit	-49000	-9000	1000
Probability	0.01	0.02	0.97

Mathematical behaviours	Marks
• completes Probability row of table correctly • completes exactly 2 entries of Profit row of table correctly • completes table correctly	1 1 1

Question 18(b) **(2 marks)**

Solution	
$E(X) = 0.01 \times (-49000) + 0.02 \times (-9000) + 0.97 \times 1000 = 300$ Hence the expected profit is $300 	
Mathematical behaviours	**Marks**
• states correct formula for $E(X)$	1
• obtains correct answer	1

Question 18(c) **(3 marks)**

Solution	
$Var(X) = (-49000-300)^2 \times 0.01 + (-9000-300)^2 \times 0.02 + (1000-300)^2 \times 0.97 = 26\,510\,000$ $Std\ Dev = \sqrt{26\,510\,000} \approx 5149$ or $E(X^2) = (-49000)^2 \times 0.01 + (-9000)^2 \times 0.02 + (1000)^2 \times 0.97 = 26600000$ $Var(X) = E(X^2) - \left(E(X)\right)^2$ $= 26600000 - (300)^2 = 26510000$ $Std\ Dev(X) = \sqrt{26510000} \approx 5149$ Note: CAS screen above shows $E(X^2) = 26600000$	
Mathematical behaviours	**Marks**
• demonstrates calculations required to obtain variance	1
• obtains variance	1
• obtains standard deviation	1

Question 19(a) **(1 mark)**

Solution	
$\int_{-1}^{2} f(x)\,dx = 5 + 16 = 21$	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 19(b) **(1 mark)**

Solution	
$\int_{-1}^{4} f(x)\,dx = 5 + 16 + 11 - 27 = 5$	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 19(c) **(1 mark)**

Solution	
$A=\int_{-1}^{4}\lvert f(x)\rvert dx=5+16+11+27=59$	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 19(d) **(2 marks)**

Solution	
Shaded area marked M $=(16\times 3)-21=27$	
Marking key/mathematical behaviours	**Marks**
• recognises area of rectangle subtract $\int_{-1}^{2} f(x)dx$ • states correct answer	1 1

Question 19(e) **(2 marks)**

(i) Correct statement is $\int_{k}^{4} f(x)dx=48$ (ii) Use CAS and solve for k : Solve $((\int_{k}^{4}(12x^2-4x^3)dx=48,k)\Rightarrow\ k=-2$	
Mathematical behaviours	**Marks**
(i) • chooses correct statement (ii) • solves for k	1 1

Question 20(a) **(3 marks)**

Solution	
For the circle, $l=2\pi r\Rightarrow r=\frac{l}{2\pi}$ $\therefore A_c=\pi\left(\frac{l}{2\pi}\right)^2$ For the square, $x=\left(\frac{100-l}{4}\right)$ $\therefore A_s=\left(\frac{100-l}{4}\right)^2$ Hence, $A=\pi\left(\frac{l}{2\pi}\right)^2+\left(\frac{100-l}{4}\right)^2$	
Mathematical behaviours	**Mark**
• demonstrates that $r=\frac{1}{2\pi}$ and states expression for the area of the circle • demonstrates that side length $=\frac{100-l}{4}$ and states expression for the area of the square • concludes formula for A	1 1 1

Question 20(b) **(5 marks)**

Solution

$A = \pi\left(\frac{l}{2\pi}\right)^2 + \left(\frac{100-l}{4}\right)^2$

$\frac{dA}{dl} = \frac{2\pi l}{4\pi^2} - \frac{1}{8}(100-l) = \frac{l}{2\pi} - \frac{1}{8}(100-l) = l\left(\frac{1}{2\pi}+\frac{1}{8}\right) - \frac{25}{2} = l\left(\frac{4+\pi}{8\pi}\right) - \frac{25}{2}$

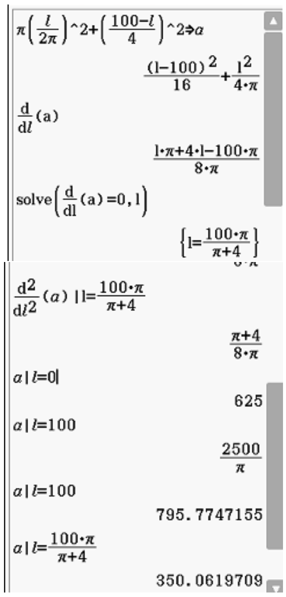
$\frac{dA}{dl} = 0 \Rightarrow l = \frac{25}{2}\left(\frac{8\pi}{4+\pi}\right) \approx 43.99\,cm$

$\frac{d^2A}{dl^2} = \frac{4+\pi}{8\pi} > 0 \Rightarrow min$

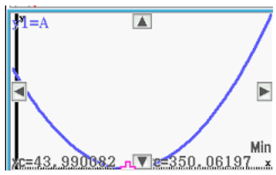
$A\big|_{l=43.99} = 350.06$

$A\big|_{l=0} = 25^2 = 625$

$A\big|_{l=100} = \pi\left(\frac{100}{2\pi}\right)^2 \approx 795.77$



Or, to establish minimum has been achieved at $l = 43.99\,cm$, states coefficient of l^2 is positive, hence minimum turning point or demonstrates with graph



Hence the minimum total area is obtained when $l = 43.99\,cm$

Mathematical behaviours	Marks
• determines $\frac{dA}{dl}$	1
• equates $\frac{dA}{dl} = 0$ and solves	1
• establishes $\frac{d^2A}{dl^2}\Big|_{l=43.99} > 0$ hence a minimum	1
• determines A for $l=0$ and $l=100$ OR demonstrates through graph or coefficient of l^2 that A is a quadratic with a minimum turning point	1
• concludes minimum area is when $l = 43.99\,cm$	1